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Flexural vibrations of poroelastic solids in the presence of static stresses

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Abstract
Employing Biot's theory of poroelasticity, three-dimensional vibrations in a poroelastic solid that is subjected to static stresses is investigated. By considering second-order coupling between stress and strain, pertinent governing equations are derived. A frequency equation is obtained in the case of static uniaxial stress or strain. Phase velocity against static uniaxial stress is computed in the case of two poroelastic solids and results are presented graphically.

Keywords
Flexural vibrations, phase velocity, poroelastic solids, static stresses, wavenumber

1. Introduction

The theory of poroelastic media originates from the requirements of specific problems of geophysics such as the problems of propagation of seismic waves. Detailed analysis of wave propagation in elastic media is given in the papers by Miklowitz (1960) and Kolsky (1963). When the body forces act on a solid, the state of the solid which is at rest gives static stress and strains. When the large static stress acts on a solid, the effect of body forces can be neglected (Mott, 1971). The static stress caused by large, externally impressed surface forces is assumed to give effects much larger than the effect of gravity. The examples of externally impressed forces are ferroelectric, piezoelectric materials which have large static stress produced in them by some external agency, such as a battery or coil (Mott, 1971). The elastic motion of an isotropic medium in the presence of body forces and static stresses is investigated (Mott, 1971). In the said paper, Mott derived the equations of motion in an elastic medium in the presence of body forces and static stresses. Further, elastic waveguide propagation in an infinite isotropic solid cylinder that is subjected to a static axial stress and strain is given in the paper (Mott, 1973). In the said paper, the effect of static axial stress and strain upon the velocity of the lowest-order flexural mode in solid circular cylinders is discussed and it has been proved that flexural waves in cylinders and transverse waves in stretched strings are of the same nature. With regard to poroelastic solids, the governing three–dimensional equations of poroelasticity are developed in the frequency domain (Biot, 1956). Various aspects of wave propagation from the perspective of geophysics and seismology are discussed (Tolstoy, 1973) in the framework of Biot's theory of poroelastic solids. Consideration can be given to flexural vibrations in poroelastic solids such as beams, plates, and shells which are three-dimensional in nature. Flexural vibrations of poroelastic plates are investigated (Theodorakopoudos and Beskos, 1994). Edge waves in poroelastic plate under plane stress conditions are studied (Reddy Perati and Tajuddin, 2003). In the said paper, the governing equations of plane stress problems in poroelastic solids are formulated. Flexural wave propagation in coated poroelastic cylinders with reference to fretting fatigue is investigated (Shah, 2011). Flexural vibrations of poroelastic circular cylindrical shells immersed in an acoustic medium are studied (Shah and Tajuddin, 2010). There is much focus on wave propagation in poroelastic cylinders and poroelastic half spaces (Tajuddin and Ahmed, 1991; Reddy Perati and Tajuddin, 2000; Reddy and Tajuddin, 2000, 2010; Shah and Tajuddin, 2010; Nageswara et al., 2011; Rajitha et al., 2012) in the framework of Biot's theory (Biot, 1956). However, the above listed publications do

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not contain investigation of flexural vibrations of poroelastic solids in the presence of static stresses. Therefore, in the present paper, this is investigated in the framework of Biot's theory (Biot, 1956). Here the phrase "poroelastic solids" in the title stands for poroelastic bars, poroelastic plates, and poroelastic shell wherein flexural vibrations occur. The pertinent governing equations are not readily available in the literature. Thus, the said equations are derived from the corresponding equations of the general three-dimensional problems. Non-dimensional phase velocity is computed as a function of static uniaxial stress and wavenumber.

The rest of the paper is organized as follows. In Section 2, formulations of governing equations are given. Equations of motion in the presence of static uniaxial stress are given in Section 3. Numerical results are described in Section 4. Finally, conclusions are given in Section 5.

2. Governing equations

Let $\vec{u}(u,v,w)$ and $\vec{U}(U,V,W)$ be the displacements of solid and fluid, respectively, in the X,Y,Z directions. Because of the second-order coupling between the stresses and strains, it is shown in the paper (Mott, 1971) that the effective stresses must be inserted in place of the usual stresses in the constitutive relations and the equations of motion. Whatever may be the boundary conditions, this effective stress must replace the usual stress at the boundary. The effective stress-strain relations are (Mott, 1971)

$$\begin{aligned}
\sigma'_{xx} &= \sigma_{xx} - \sigma_{xy}\frac{\partial u}{\partial y} - \sigma_{xz}\frac{\partial u}{\partial z}\\
\sigma'_{xy} &= \sigma_{xy} - \sigma_{xx}\frac{\partial v}{\partial x} - \sigma_{xz}\frac{\partial v}{\partial z}\\
\sigma'_{xz} &= \sigma_{xz} - \sigma_{xx}\frac{\partial w}{\partial x} - \sigma_{xy}\frac{\partial w}{\partial y}\\
\sigma'_{yy} &= \sigma_{yy} - \sigma_{yx}\frac{\partial v}{\partial x} - \sigma_{yz}\frac{\partial v}{\partial z}\\
\sigma'_{yx} &= \sigma_{yx} - \sigma_{yy}\frac{\partial u}{\partial y} - \sigma_{yz}\frac{\partial u}{\partial z}\\
\sigma'_{yz} &= \sigma_{yz} - \sigma_{yx}\frac{\partial w}{\partial x} - \sigma_{yy}\frac{\partial w}{\partial y}\\
\sigma'_{zx} &= \sigma_{zx} - \sigma_{zy}\frac{\partial u}{\partial y} - \sigma_{zz}\frac{\partial u}{\partial z}\\
\sigma'_{zy} &= \sigma_{zy} - \sigma_{zx}\frac{\partial v}{\partial x} - \sigma_{zz}\frac{\partial v}{\partial z}\\
\sigma'_{zz} &= \sigma_{zz} - \sigma_{zx}\frac{\partial w}{\partial x} - \sigma_{zy}\frac{\partial w}{\partial y}
\end{aligned} \tag{1}$$

As there are no shear components for the fluid pressure, there will not be any effective fluid pressure as that of solid. The expression for the fluid pressure remains the same. In all the above, usual stresses σ_{ij} and fluid pressure s are (Biot, 1956)

$$\begin{aligned}
\sigma_{ij} &= 2Ne_{ij} + (Ae + Q\varepsilon)\delta_{ij} \quad (i,j = x,y,z),\\
s &= Qe + R\varepsilon.
\end{aligned} \tag{2}$$

In equation (2), e_{ij}'s are strain displacements, δ_{ij} is the well-known Kronecker delta function, e and ε are the dilatations of solid and fluid, respectively, and A,N,Q,R are all poroelastic constants. Substitution of effective stresses for usual stresses, the equations of motion appear in the following manner:

$$\begin{aligned}
&\frac{\partial}{\partial x}\left(\sigma'_{xx}\left(1+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\right)+\frac{\partial}{\partial y}\left(\sigma'_{xy}\left(1+\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right)\right)\\
&\quad+\frac{\partial}{\partial z}\left(\sigma'_{xz}\left(1+\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right)+F_x=\frac{\partial^2}{\partial t^2}(\rho_{11}u+\rho_{12}U),\\
&\frac{\partial}{\partial x}\left(\sigma'_{yx}\left(1+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\right)+\frac{\partial}{\partial y}\left(\sigma'_{yy}\left(1+\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right)\right)\\
&\quad+\frac{\partial}{\partial z}\left(\sigma'_{yz}\left(1+\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right)+F_y=\frac{\partial^2}{\partial t^2}(\rho_{11}v+\rho_{12}V),\\
&\frac{\partial}{\partial x}\left(\sigma'_{zx}\left(1+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\right)+\frac{\partial}{\partial y}\left(\sigma'_{zy}\left(1+\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right)\right)\\
&\quad+\frac{\partial}{\partial z}\left(\sigma'_{zz}\left(1+\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right)+F_z=\frac{\partial^2}{\partial t^2}(\rho_{11}w+\rho_{12}W),\\
&Q\frac{\partial e}{\partial x}+R\frac{\partial \varepsilon}{\partial x}=\frac{\partial^2}{\partial t^2}(\rho_{12}u+\rho_{22}U),\\
&Q\frac{\partial e}{\partial y}+R\frac{\partial \varepsilon}{\partial y}=\frac{\partial^2}{\partial t^2}(\rho_{12}v+\rho_{22}V),\\
&Q\frac{\partial e}{\partial z}+R\frac{\partial \varepsilon}{\partial z}=\frac{\partial^2}{\partial t^2}(\rho_{12}w+\rho_{22}W).
\end{aligned} \tag{3}$$

In equation (3), $\sigma'_{ij}(i=j=x,y,z)$ are the effective stresses, ρ_{ij} are mass coefficients such that the sums $\rho_{11}+\rho_{12}$ and $\rho_{12}+\rho_{22}$ represents the mass of solid and fluid per unit volume of bulk material following (Biot, 1956). The coefficient ρ_{12} is a mass coupling parameter between solid and fluid phases. Furthermore, the mass parameters obey the inequalities $\rho_{11}>0$, $\rho_{12}>0$, $\rho_{22}>0$ $(\rho_{11}\rho_{22}-\rho_{12}^2)>0$ and F_x, F_y, and F_z are the components of the body force vector $\vec{F}$. Equations (3) together with equations (1) are to be satisfied at every interior point of the body, and on the surface of the body. Appropriate boundary conditions are to be imposed so that problem can be solved completely. Thus, in the presence of static stress, the solution of a dynamical problem in poroelastic solids can be completely determined. By neglecting the liquid

effects, that is, setting $\rho_{12}=0$, $\rho_{22}=0$, $\rho_{11}=\rho$, $A-\frac{Q^2}{R}=\lambda$, $N=\mu$, the results of purely elastic solid (Mott, 1971) are recovered as a special case. In the above, λ, μ, and ρ are Lame constants, and density of purely elastic solid, respectively. To construct the relationships between the time variant and the static poroelastic quantities, the particle displacements and the stresses are written in the following form (Mott, 1971):

$$\begin{aligned}
u &= \sum u_n(x,y,z,\omega_n t) = u_0+u_1+u_2+\cdots,\\
v &= \sum v_n(x,y,z,\omega_n t) = v_0+v_1+v_2+\cdots,\\
w &= \sum w_n(x,y,z,\omega_n t) = w_0+w_1+w_2+\cdots,\\
U &= \sum U_n(x,y,z,\omega_n t) = U_0+U_1+U_2+\cdots,\\
V &= \sum V_n(x,y,z,\omega_n t) = V_0+V_1+V_2+\cdots,\\
W &= \sum W_n(x,y,z,\omega_n t) = W_0+W_1+W_2+\cdots,\\
\sigma_{ij} &= \sum \sigma_{ij_n}(x,y,z,\omega_n t) = \sigma_{ij_0}+\sigma_{ij_1}+\sigma_{ij_2}+\cdots,
\end{aligned} \tag{4}$$

where ω_n is the nth angular frequency. Using equations (1), (2) and (4) in the equations (3), there follows the static equations ($n=0$) given below:

$$\begin{aligned}
&\frac{\partial}{\partial x}\left(\left(\sigma_{xx_0}-\sigma_{xy_0}\frac{\partial u_0}{\partial y}-\sigma_{xz_0}\frac{\partial u_0}{\partial z}\right)\left(1+\frac{\partial v_0}{\partial y}+\frac{\partial w_0}{\partial z}\right)\right)\\
&\quad+\frac{\partial}{\partial y}\left(\left(\sigma_{xy_0}-\sigma_{xx_0}\frac{\partial v_0}{\partial x}-\sigma_{xz_0}\frac{\partial v_0}{\partial z}\right)\left(1+\frac{\partial u_0}{\partial x}+\frac{\partial w_0}{\partial z}\right)\right)\\
&\quad+\frac{\partial}{\partial z}\left(\left(\sigma_{xz_0}-\sigma_{xx_0}\frac{\partial w_0}{\partial x}-\sigma_{xy_0}\frac{\partial w_0}{\partial y}\right)\left(1+\frac{\partial u_0}{\partial x}+\frac{\partial v_0}{\partial y}\right)\right)\\
&\quad+F_{x_0}=0,\\
&\frac{\partial}{\partial x}\left(\left(\sigma_{yx_0}-\sigma_{yy_o}\frac{\partial u_0}{\partial y}-\sigma_{yz_0}\frac{\partial u_0}{\partial z}\right)\left(1+\frac{\partial v_0}{\partial y}+\frac{\partial w_0}{\partial z}\right)\right)\\
&\quad+\frac{\partial}{\partial y}\left(\left(\sigma_{yy_0}-\sigma_{yx_0}\frac{\partial v_0}{\partial x}-\sigma_{yz_0}\frac{\partial v_0}{\partial z}\right)\left(1+\frac{\partial u_0}{\partial x}+\frac{\partial w_0}{\partial z}\right)\right)\\
&\quad+\frac{\partial}{\partial z}\left(\left(\sigma_{yz_0}-\sigma_{yy_0}\frac{\partial w_0}{\partial y}-\sigma_{yx_0}\frac{\partial w_0}{\partial x}\right)\left(1+\frac{\partial u_0}{\partial x}+\frac{\partial v_0}{\partial y}\right)\right)\\
&\quad+F_{y_0}=0,\\
&\frac{\partial}{\partial x}\left(\left(\sigma_{zx_0}-\sigma_{zz_0}\frac{\partial u_0}{\partial z}-\sigma_{zy_0}\frac{\partial u_0}{\partial y}\right)\left(1+\frac{\partial v_0}{\partial y}+\frac{\partial w_0}{\partial z}\right)\right)\\
&\quad+\frac{\partial}{\partial y}\left(\left(\sigma_{zy_0}-\sigma_{zz_0}\frac{\partial v_0}{\partial z}-\sigma_{zx_0}\frac{\partial v_0}{\partial x}\right)\left(1+\frac{\partial u_0}{\partial x}+\frac{\partial w_0}{\partial z}\right)\right)\\
&\quad+\frac{\partial}{\partial z}\left(\left(\sigma_{zz_0}-\sigma_{zy_0}\frac{\partial w_0}{\partial y}-\sigma_{zx_0}\frac{\partial w_0}{\partial x}\right)\left(1+\frac{\partial v_0}{\partial y}+\frac{\partial u_0}{\partial x}\right)\right)\\
&\quad+F_{z_0}=0,
\end{aligned}$$

$$\begin{aligned}
&Q\frac{\partial e}{\partial x}+R\frac{\partial \varepsilon}{\partial x}=0,\\
&Q\frac{\partial e}{\partial y}+R\frac{\partial \varepsilon}{\partial y}=0,\\
&Q\frac{\partial e}{\partial z}+R\frac{\partial \varepsilon}{\partial z}=0.
\end{aligned} \tag{5}$$

In the above, $e=\frac{\partial u_0}{\partial x}+\frac{\partial v_0}{\partial y}+\frac{\partial w_0}{\partial z}$ and $\varepsilon=\frac{\partial U_0}{\partial x}+\frac{\partial V_0}{\partial y}+\frac{\partial W_0}{\partial z}$.

Similarly, the harmonic equations ($n=1$) are obtained, which are:

$$\begin{aligned}
&\frac{\partial}{\partial x}\left(\left(\sigma_{xx_1}-\sigma_{xy_1}\frac{\partial u_0}{\partial y}-\sigma_{xy_0}\frac{\partial u_1}{\partial y}-\sigma_{xz_1}\frac{\partial u_0}{\partial z}-\sigma_{xz_0}\frac{\partial u_1}{\partial z}\right)\right.\\
&\quad\times\left(1+\frac{\partial v_0}{\partial y}+\frac{\partial w_0}{\partial z}\right)+\left(\sigma_{xx_0}-\sigma_{xy_0}\frac{\partial u_0}{\partial y}-\sigma_{xz_0}\frac{\partial u_0}{\partial z}\right)\\
&\quad\left.\times\left(\frac{\partial v_1}{\partial y}+\frac{\partial w_1}{\partial z}\right)\right)+\frac{\partial}{\partial y}\left(\left(\sigma_{xy_1}-\sigma_{xx_1}\frac{\partial v_0}{\partial x}-\sigma_{xx_0}\frac{\partial v_1}{\partial x}\right.\right.\\
&\quad\left.-\sigma_{xz_1}\frac{\partial v_0}{\partial z}-\sigma_{xz_0}\frac{\partial v_1}{\partial z}\right)\left(1+\frac{\partial u_0}{\partial x}+\frac{\partial w_0}{\partial z}\right)\\
&\quad\left.+\left(\sigma_{xy_0}-\sigma_{xx_0}\frac{\partial v_0}{\partial x}-\sigma_{xz_0}\frac{\partial v_0}{\partial z}\right)\left(\frac{\partial u_1}{\partial x}+\frac{\partial w_1}{\partial z}\right)\right)\\
&\quad+\frac{\partial}{\partial z}\left(\left(\sigma_{xz_1}-\sigma_{xx_1}\frac{\partial w_0}{\partial x}-\sigma_{xx_0}\frac{\partial w_1}{\partial x}-\sigma_{xy_1}\frac{\partial w_0}{\partial y}-\sigma_{xy_0}\frac{\partial w_1}{\partial y}\right)\right.\\
&\quad\times\left(1+\frac{\partial u_0}{\partial x}+\frac{\partial v_0}{\partial y}\right)+\left(\sigma_{xz_0}-\sigma_{xx_0}\frac{\partial w_0}{\partial x}-\sigma_{xy_0}\frac{\partial w_0}{\partial y}\right)\\
&\quad\left.\times\left(\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y}\right)\right)+F_{x_1}=\frac{\partial^2}{\partial t^2}(\rho_{11}u_1+\rho_{12}U_1),\\
&\frac{\partial}{\partial x}\left(\left(\sigma_{xy_1}-\sigma_{yy_1}\frac{\partial u_0}{\partial y}-\sigma_{yy_0}\frac{\partial u_1}{\partial y}-\sigma_{yz_1}\frac{\partial u_0}{\partial z}-\sigma_{yz_0}\frac{\partial u_1}{\partial z}\right)\right.\\
&\quad\times\left(1+\frac{\partial v_0}{\partial y}+\frac{\partial w_0}{\partial z}\right)+\left(\sigma_{xy_0}-\sigma_{yy_0}\frac{\partial u_0}{\partial y}-\sigma_{yz_0}\frac{\partial u_0}{\partial z}\right)\\
&\quad\left.\times\left(\frac{\partial v_1}{\partial y}+\frac{\partial w_1}{\partial z}\right)\right)+\frac{\partial}{\partial y}\left(\left(\sigma_{yy_1}-\sigma_{yx_1}\frac{\partial v_0}{\partial x}-\sigma_{yx_0}\frac{\partial v_1}{\partial x}-\sigma_{yz_1}\frac{\partial v_0}{\partial z}\right.\right.\\
&\quad\left.-\sigma_{yz_0}\frac{\partial v_1}{\partial z}\right)\left(1+\frac{\partial u_0}{\partial x}+\frac{\partial w_0}{\partial z}\right)+\left(\sigma_{yy_0}-\sigma_{yx_0}\frac{\partial v_0}{\partial x}-\sigma_{yz_0}\frac{\partial v_0}{\partial z}\right)\\
&\quad\left.\times\left(\frac{\partial u_1}{\partial x}+\frac{\partial w_1}{\partial z}\right)\right)+\frac{\partial}{\partial z}\left(\left(\sigma_{yz_1}-\sigma_{yy_1}\frac{\partial w_0}{\partial y}-\sigma_{yy_0}\frac{\partial w_1}{\partial y}\right.\right.\\
&\quad\left.-\sigma_{yx_1}\frac{\partial w_0}{\partial x}-\sigma_{yx_0}\frac{\partial w_1}{\partial x}\right)\left(1+\frac{\partial u_0}{\partial x}+\frac{\partial v_0}{\partial y}\right)\\
&\quad\left.+\left(\sigma_{yz_0}-\sigma_{yy_0}\frac{\partial w_0}{\partial y}-\sigma_{yx_0}\frac{\partial w_0}{\partial x}\right)\left(\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y}\right)\right)+F_{y_1}\\
&=\frac{\partial^2}{\partial t^2}(\rho_{11}v_1+\rho_{12}V_1),
\end{aligned}$$

$$\frac{\partial}{\partial x}\left(\left(\sigma_{zx_1}-\sigma_{zz_1}\frac{\partial u_0}{\partial z}-\sigma_{zz_0}\frac{\partial u_1}{\partial z}-\sigma_{zy_1}\frac{\partial u_0}{\partial y}-\sigma_{zy_0}\frac{\partial u_1}{\partial y}\right)\right.$$
$$\times\left(1+\frac{\partial v_0}{\partial y}+\frac{\partial w_0}{\partial z}\right)+\left(\sigma_{zx_0}-\sigma_{zz_0}\frac{\partial u_0}{\partial z}-\sigma_{zy_0}\frac{\partial u_0}{\partial y}\right)$$
$$\left.\times\left(\frac{\partial w_1}{\partial z}+\frac{\partial v_1}{\partial y}\right)\right)+\frac{\partial}{\partial y}\left(\left((\sigma_{zy_1}-\sigma_{zz_1}\frac{\partial v_0}{\partial z}-\sigma_{zz_0}\frac{\partial v_1}{\partial z}\right.\right.$$
$$-\sigma_{zx_1}\frac{\partial v_0}{\partial x}-\sigma_{zx_0}\frac{\partial v_1}{\partial x}\left(1+\frac{\partial u_0}{\partial x}+\frac{\partial w_0}{\partial z}\right)$$
$$\left.+\left(\sigma_{zy_0}-\sigma_{zz_0}\frac{\partial v_0}{\partial z}-\sigma_{zx_0}\frac{\partial v_0}{\partial x}\right)\left(\frac{\partial u_1}{\partial x}+\frac{\partial w_1}{\partial z}\right)\right)$$
$$+\frac{\partial}{\partial z}\left(\left(\sigma_{zz_1}-\sigma_{zy_1}\frac{\partial w_0}{\partial y}-\sigma_{zy_0}\frac{\partial w_1}{\partial y}-\sigma_{zx_1}\frac{\partial w_0}{\partial x}-\sigma_{zx_0}\frac{\partial w_1}{\partial x}\right)\right.$$
$$\times\left(1+\frac{\partial u_0}{\partial x}+\frac{\partial v_0}{\partial y}\right)+\left(\sigma_{zz_0}-\sigma_{zy_0}\frac{\partial w_0}{\partial y}-\sigma_{zx_0}\frac{\partial w_0}{\partial x}\right)$$
$$\left.\times\left(\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y}\right)\right)+F_{z_1}=\frac{\partial^2}{\partial t^2}(\rho_{11}w_1+\rho_{12}W_1),$$
$$Q\frac{\partial e}{\partial x}+R\frac{\partial \varepsilon}{\partial x}=\frac{\partial^2}{\partial t^2}(\rho_{12}u_1+\rho_{22}U_1),$$
$$Q\frac{\partial e}{\partial y}+R\frac{\partial \varepsilon}{\partial y}=\frac{\partial^2}{\partial t^2}(\rho_{12}v_1+\rho_{22}V_1),$$
$$Q\frac{\partial e}{\partial z}+R\frac{\partial \varepsilon}{\partial z}=\frac{\partial^2}{\partial t^2}(\rho_{12}w_1+\rho_{22}W_1). \tag{6}$$

In all the above, $e=\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y}+\frac{\partial w_1}{\partial z}$, $\varepsilon=\frac{\partial U_1}{\partial x}+\frac{\partial V_1}{\partial y}+\frac{\partial W_1}{\partial z}$, and $F_{x_i}, F_{y_i}, F_{z_i}$ are the body forces which have the frequency ω. Equations (5) and (6) are static equations, and first harmonic equations, respectively.

3. Equations of motion in presence of static uniaxial stress

Assume that the static uniaxial applied stress is acting in the direction of z axis. Then there follows

$$\begin{aligned}\sigma_{ij_0}&=e_{ij_0}=0,\quad (i\neq j),\\ \sigma_{xx_0}&=\sigma_{yy_0}=0,\\ F_{x_0}&=F_{y_0}=F_{z_0}=0,\end{aligned} \tag{7}$$

σ_{zz_0} is the applied uniform static uniaxial stress. These conditions are applicable for a bar with a uniform cross section and with its side surface stress free. Substituting equations (7) in equations (5), it can be seen that equations (5) are automatically satisfied and the static strains are found to be $e_{xx_0}=e_{yy_0}=-\vartheta e_{zz_0}$.

Substituting these equations in the first equation of equation (2), the following equation is obtained:

$$e_{zz_0}=\frac{\sigma_{zz_0}-Q\varepsilon}{Y}. \tag{8}$$

In the above, $\vartheta=\frac{A}{2(A+N)}$ is Poisson ratio, and $Y=\frac{N(P+2A)}{A+N}$ is Young's modulus.

Substituting equation (7) and equation (8) in equation (6), the equations of motion for the first harmonic terms are obtained and are:

$$N\nabla^2u+(A+N)\frac{\partial e}{\partial x}+Q\frac{\partial\varepsilon}{\partial x}+\frac{1}{2(A+N)Y}$$
$$\times\left(\sigma_{zz_0}-Q\left(\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}+\frac{\partial W}{\partial z}\right)\right)$$
$$\times\left(p\left(2N\frac{\partial^2u}{\partial x^2}+A\frac{\partial e}{\partial x}+Q\frac{\partial\varepsilon}{\partial x}+N\frac{\partial^2u}{\partial y^2}+N\frac{\partial^2v}{\partial x\partial y}\right)\right.$$
$$\left.-2AN\left(\frac{\partial^2w}{\partial x\partial z}+\frac{\partial^2u}{\partial z^2}\right)\right)=\frac{\partial^2}{\partial t^2}(\rho_{11}u+\rho_{12}U),$$

$$N\nabla^2v+(A+N)\frac{\partial e}{\partial y}+Q\frac{\partial\varepsilon}{\partial y}+\frac{1}{2(A+N)Y}$$
$$\times\left(\sigma_{zz_0}-Q\left(\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}+\frac{\partial W}{\partial z}\right)\right)$$
$$\times\left(P\left(2N\frac{\partial^2v}{\partial y^2}+A\frac{\partial e}{\partial y}+Q\frac{\partial\varepsilon}{\partial y}+N\frac{\partial^2v}{\partial x^2}+N\frac{\partial^2u}{\partial x\partial y}\right)\right.$$
$$\left.-2AN\left(\frac{\partial^2w}{\partial y\partial z}+\frac{\partial^2v}{\partial z^2}\right)\right)=\frac{\partial^2}{\partial t^2}(\rho_{11}v+\rho_{12}V),$$
$$N\nabla^2w+(A+N)\frac{\partial e}{\partial z}+Q\frac{\partial\varepsilon}{\partial z}+\frac{1}{2(A+N)Y}$$
$$\times\left(\sigma_{zz_0}-Q\left(\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}+\frac{\partial W}{\partial z}\right)\right)$$
$$\times\left(P\left(N\frac{\partial^2w}{\partial x^2}+N\frac{\partial^2w}{\partial y^2}+N\frac{\partial^2u}{\partial x\partial z}+N\frac{\partial^2v}{\partial y\partial z}\right.\right.$$
$$\left.\left.-\sigma_{zz_0}\left(\frac{\partial^2u}{\partial x\partial z}+\frac{\partial^2v}{\partial y\partial z}\right)\right)-2A\left(2N\frac{\partial^2w}{\partial z^2}+A\frac{\partial e}{\partial z}+Q\frac{\partial\varepsilon}{\partial z}\right)\right)$$
$$=\frac{\partial^2}{\partial t^2}(\rho_{11}w+\rho_{12}W),$$
$$Q\frac{\partial e}{\partial x}+R\frac{\partial\varepsilon}{\partial x}=\frac{\partial^2}{\partial t^2}(\rho_{12}u+\rho_{22}U),$$
$$Q\frac{\partial e}{\partial y}+R\frac{\partial\varepsilon}{\partial y}=\frac{\partial^2}{\partial t^2}(\rho_{12}v+\rho_{22}V),$$
$$Q\frac{\partial e}{\partial z}+R\frac{\partial\varepsilon}{\partial z}=\frac{\partial^2}{\partial t^2}(\rho_{12}w+\rho_{22}W). \tag{9}$$

In equations (9), the suffix one on displacement components and the stresses has been dropped, and $F_{x_1}=F_{y_1}=F_{z_1}=0$. Due to the assumption of infinitesimal deformation in the linear theory of elasticity, the product terms can be neglected and the equations of motion in this case are obtained as follows:

$$N\nabla^2u+(A+N)\frac{\partial e}{\partial x}+Q\frac{\partial\varepsilon}{\partial x}+\frac{1}{2(A+N)}\frac{\sigma_{zz_0}}{Y}$$

$$\times\left(P\left(2N\frac{\partial^2 u}{\partial x^2}+A\frac{\partial e}{\partial x}+Q\frac{\partial \varepsilon}{\partial x}+N\frac{\partial^2 u}{\partial y^2}+N\frac{\partial^2 v}{\partial x\partial y}\right)\right.$$
$$\left.-2AN\left(\frac{\partial^2 w}{\partial x\partial z}+\frac{\partial^2 u}{\partial z^2}\right)\right)=\frac{\partial^2}{\partial t^2}(\rho_{11}u+\rho_{12}U),$$
$$N\nabla^2 v+(A+N)\frac{\partial e}{\partial y}+Q\frac{\partial \varepsilon}{\partial y}+\frac{1}{2(A+N)}\frac{\sigma_{zz_0}}{Y}$$
$$\times\left(P\left(2N\frac{\partial^2 v}{\partial y^2}+A\frac{\partial e}{\partial y}+Q\frac{\partial \varepsilon}{\partial y}+N\frac{\partial^2 v}{\partial x^2}+N\frac{\partial^2 u}{\partial x\partial y}\right)\right.$$
$$\left.-2AN\left(\frac{\partial^2 w}{\partial y\partial z}+\frac{\partial^2 v}{\partial z^2}\right)\right)=\frac{\partial^2}{\partial t^2}(\rho_{11}v+\rho_{12}V),$$
$$N\nabla^2 w+(A+N)\frac{\partial e}{\partial z}+Q\frac{\partial \varepsilon}{\partial z}+\frac{1}{2(A+N)}\frac{\sigma_{zz_0}}{Y}$$
$$\times\left(P\left(N\frac{\partial^2 w}{\partial x^2}+N\frac{\partial^2 w}{\partial y^2}+N\frac{\partial^2 u}{\partial x\partial z}+N\frac{\partial^2 v}{\partial y\partial z}\right.\right.$$
$$\left.\left.-\sigma_{zz_0}\left(\frac{\partial^2 u}{\partial x\partial z}+\frac{\partial^2 v}{\partial y\partial z}\right)\right)-2A\left(2N\frac{\partial^2 w}{\partial z^2}+A\frac{\partial e}{\partial z}+Q\frac{\partial \varepsilon}{\partial z}\right)\right)$$
$$=\frac{\partial^2}{\partial t^2}(\rho_{11}w+\rho_{12}W),$$
$$Q\frac{\partial e}{\partial x}+R\frac{\partial \varepsilon}{\partial x}=\frac{\partial^2}{\partial t^2}(\rho_{12}u+\rho_{22}U),$$
$$Q\frac{\partial e}{\partial y}+R\frac{\partial \varepsilon}{\partial y}=\frac{\partial^2}{\partial t^2}(\rho_{12}v+\rho_{22}V),$$
$$Q\frac{\partial e}{\partial z}+R\frac{\partial \varepsilon}{\partial z}=\frac{\partial^2}{\partial t^2}(\rho_{12}w+\rho_{22}W). \tag{10}$$

Now, it can be assumed the solution to eqations (10) in the following form:

$$\begin{aligned}
u(x,y,z)&=C_1\,e^{j\omega t-j(k_1x+k_2y+k_3z)},\\
v(x,y,z)&=C_2\,e^{j\omega t-j(k_1x+k_2y+k_3z)},\\
w(x,y,z)&=C_3\,e^{j\omega t-j(k_1x+k_2y+k_3z)},\\
U(x,y,z)&=C_4\,e^{j\omega t-j(k_1x+k_2y+k_3z)},\\
V(x,y,z)&=C_5\,e^{j\omega t-j(k_1x+k_2y+k_3z)},\\
W(x,y,z)&=C_6\,e^{j\omega t-j(k_1x+k_2y+k_3z)}.
\end{aligned} \tag{11}$$

In all the above, $C_1, C_2, C_3, C_4, C_5, C_6$ are arbitrary constants, j is the complex unity, and $k_i\,(i=1,2,3)$ is the wavenumber in the i th direction such that the wavenumber $k=\sqrt{k_1^2+k_2^2+k_3^2}$. Substituting equation (11) in equation (10), the equations of motion in terms of displacements are as follows:

$$\left(Pk_1^2+N\left(k_2^2+k_3^2\right)+\frac{P}{2(A+N)}\frac{\sigma_{zz_0}}{Y}\left(Pk_1^2+Nk_2^2-2ANk_3^2\right)\right.$$
$$\left.-\rho_{11}\omega^2\right)C_1+\left((A+N)+\frac{P}{2(A+N)}\frac{\sigma_{zz_0}}{Y}(A+Q+N)\right)$$
$$\times k_1k_2C_2+\left((A+N)+\frac{P}{2(A+N)}\frac{\sigma_{zz_0}}{Y}(A(1-2N)\right)\Bigg)$$
$$\times k_1k_3C_3+\left(\left(1+\frac{P}{2(A+N)}\frac{\sigma_{zz_0}}{Y}\right)Qk_1^2-\rho_{12}\omega^2\right)C_4$$
$$+\left(\left(1+\frac{P}{2(A+N)}\frac{\sigma_{zz_0}}{Y}\right)Qk_1k_2\right)C_5$$
$$+\left(\left(1+\frac{P}{2(A+N)}\frac{\sigma_{zz_0}}{Y}\right)Qk_1k_3\right)C_6=0,$$
$$\left((A+N)+\frac{1}{2(A+N)}\frac{\sigma_{zz_0}}{Y}(P(A+N))\right)k_1k_2C_1$$
$$+\left(N\left(k_1^2+k_2^2+k_3^2\right)+(A+N)k_2^2+\frac{1}{2(A+N)}\right.$$
$$\times\frac{\sigma_{zz_0}}{Y}(P\left(Pk_2^2+Nk_1^2\right)-2ANk_3^2)-\rho_{11}\omega^2)C_2+\Bigg((A+N)$$
$$\left.+\frac{1}{2(A+N)}\frac{\sigma_{zz_0}}{Y}(PA-2AN)\right)k_2k_3C_3$$
$$+\left(1+\frac{P}{2(A+N)}\frac{\sigma_{zz_0}}{Y}\right)Qk_1k_2\Bigg)C_4$$
$$+\left(Qk_2^2(1+\frac{P}{2(A+N)}\frac{\sigma_{zz_0}}{Y}\right)-\rho_{12}\omega^2\Bigg)C_5$$
$$+\left(Qk_2k_3(1+\frac{P}{2(A+N)}\frac{\sigma_{zz_0}}{Y})\right)\Bigg)C_6=0,$$
$$\left((A+N)+\frac{1}{2(A+N)}\frac{\sigma_{zz_0}}{Y}\left(PN-\sigma_{zz_0}-2A^2\right)\right)k_1k_3C_1$$
$$+\left((A+N)+\frac{1}{2(A+N)}\frac{\sigma_{zz_0}}{Y}\left(PN-\sigma_{zz_0}-2A^2\right)\right)k_2k_3C_2$$
$$+\left(N\left(k_1^2+k_2^2+k_3^2\right)+(A+N)k_3^2+\frac{1}{2(A+N)}\right.$$
$$\times\frac{\sigma_{zz_0}}{Y}\left(PN\left(k_1^2+k_2^2\right)-4ANk_3^2-2A^2k_3^2-\rho_{11}\omega^2\right)C_3$$
$$+\left(Q(k_1k_3)\left(1-\frac{2A}{2(A+N)}\frac{\sigma_{zz_0}}{Y}\right)C_4\right.$$
$$+\left(Q(k_2k_3)\left(1-\frac{2A}{2(A+N)}\frac{\sigma_{zz_0}}{Y}\right)\right)C_5$$
$$+\left(Qk_3^2\left(1-\frac{2A}{2(A+N)}\frac{\sigma_{zz_0}}{Y}\right)-\rho_{12}\omega^2\right)C_6=0,$$
$$\left(Qk_1^2-\rho_{12}\omega^2\right)C_1+Q(k_1k_2C_2+k_1k_3C_3)$$
$$+\left(Rk_1^2-\rho_{22}\omega^2\right)C_4+R(k_1k_2C_5+k_1k_3C_6)=0,$$
$$Qk_1k_2C_1+\left(Qk_2^2-\rho_{12}\omega^2\right)C_2+Qk_1k_3C_3$$
$$+Rk_1k_2C_4+\left(Rk_2^2-\rho_{22}\omega^2\right)C_5+Rk_2k_3C_6=0,$$
$$Q(k_1k_3C_1+k_2k_3C_2)+\left(Qk_3^2-\rho_{12}\omega^2\right)C_3$$
$$+R(k_1k_3C_4+k_2k_3C_5)+\left(Rk_3^2-\rho_{22}\omega^2\right)C_6=0. \tag{12}$$

4. Numerical results

For the numerical work, the propagation is considered along z direction. In this case, $k_1=k_2=0$, and equations (12) reduce to the following

Table 1. Material parameters.

Material parameters	a_1	a_2	a_3	a_4	d_1	d_2	d_3	$\tilde{z}$
Material-1	0.843	0.65	0.28	0.234	0.901	−0.001	0.101	3.851
Material-2	0.96	0.006	0.028	0.412	0.887	0	0.123	2.129

matrix form:

$$\begin{bmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = 0, \tag{13}$$

where

$$A_{11} = \left(Nk_3^2 - \frac{PA}{(P+2A)} k_3^2 \sigma_{zz_0} \right) - NV_s^{-2} \omega^2,$$

$$A_{22} = \left(Nk_3^2 - \frac{PA}{(P+2A)} k_3^2 \sigma_{zz_0} \right) - NV_s^{-2} \omega^2,$$

$$A_{33} = \left(Pk_3^2 + \frac{PA}{N(P+2A)} k_3^2 \sigma_{zz_0} - \rho_{11}\omega^2 \right) - \left(Qk_3^2 \left(1 - \frac{A}{N(P+2A)} \sigma_{zz_0} \right) - \rho_{12}\omega^2 \right) \times \left(\frac{Qk_3^2 - \rho_{12}\omega^2}{Rk_3^2 - \rho_{22}\omega^2} \right),$$

and $V_S^2 = \frac{N\rho_{22}}{\rho_{11}\rho_{22} - \rho_{12}^2}$ is the shear wave velocity.

For a nontrivial solution, the determinant of coefficients matrix is zero. Accordingly, the following frequency equation can be obtained:

$$\begin{vmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & A_{33} \end{vmatrix} = 0. \tag{14}$$

The frequency equation is investigated by introducing the non-dimensional quantities given below:

$$a_1 = \frac{P}{H}, \quad a_2 = \frac{Q}{H}, \quad a_3 = \frac{R}{H}, \quad a_4 = \frac{N}{H},$$
$$d_1 = \frac{\rho_{11}}{\rho}, \quad d_2 = \frac{\rho_{12}}{\rho}, \quad d_3 = \frac{\rho_{22}}{\rho}, \; \tilde{z} = \left(\frac{V_0}{V_s} \right)^2,$$
$$\rho = \rho_{11} + 2\rho_{12} + \rho_{22}, \quad H = P + 2Q + R, \quad V_0^2 = \frac{H}{\rho},$$
$$m = \frac{c}{c_0}, \quad c = \frac{\omega}{k}, \quad c_0^2 = \frac{N}{\rho}. \tag{15}$$

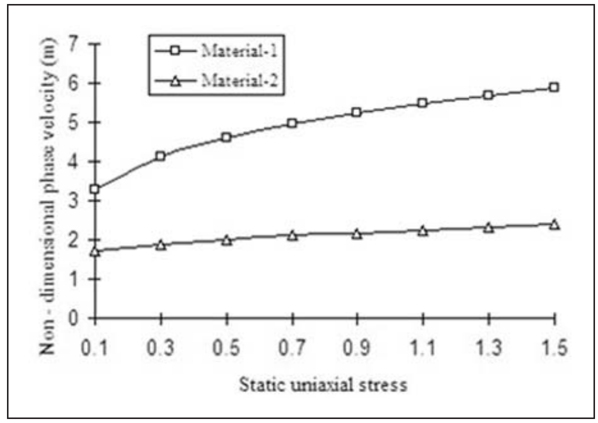

Figure 1. Variation of non-dimensional phase velocity with the static uniaxial stress.

In equation (15), c is phase velocity, and m is the non-dimensional phase velocity. Employing these non-dimensional quantities in the frequency equation, an implicit relation between non-dimensional phase velocity (m), static uniaxial stress and wavenumber is obtained. For the numerical process, two types of materials are considered, namely, material-1 is made up of sandstone saturated with kerosene (Yew and Jogi, 1976), while material-2 is sandstone saturated with water (Fatt,1959). The physical parameters of the said materials pertaining to equation (15) are given in Table 1. Static uniaxial stresses under consideration are external surface forces which can vary, hence phase velocity values are computed against static uniaxial stress for a fixed wavenumber. The values are computed using the bisection method implemented in MATLAB and the results are depicted in Figure 1. Figure 1 shows the plots of non-dimensional phase velocity against the static uniaxial stress for fixed wavenumber in the case of material-1 and material-2. From this figure it is clear that material-1 value is greater than that of material-2 for the same value of applied static stress. Both the materials are sandstone related and differ in only the fluid part. Hence, it can be inferred that the fluid part is causing above discrepancy. When the static stress increases, non-dimensional phase velocity values increase for both materials. Moreover, for fixed static uniaxial stress, the non-dimensional phase velocity value is computed against

wavenumber, and the values found to be the same for all values of wavenumber. For example, when the static stress is 0.3, the non-dimensional phase velocity is 4.8047 for all the values of the wavenumber in the range 1–20. Therefore, in this case, it may be concluded that phase velocity is independent of wavenumber.

5. Conclusions

Employing Biot's theory, flexural vibrations of poroelastic solids in the presence of static stress are investigated (Biot, 1956). Pertinent constitutive relations and the equations of motion are derived. Non-dimensional phase velocity against static uniaxial stress for fixed wavenumber is computed for two types of poroelastic solids. Static uniaxial stresses under consideration are external surface forces which can vary. Therefore, the phase velocity is computed against static uniaxial stress. From the results, it is clear that material-1 value is much greater than that of material-2 for the same value of applied static stress. Both the materials are sandstone related and differ in only the fluid part. Hence, it can be inferred that the fluid part is causing the above discrepancy. When the uniaxial static stress is fixed, it is concluded that the phase velocity is independent of wavenumber. Similar analysis can be made for different poroelastic solids if the values of their poroelastic constants are available.

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